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Susy, Gauss, Heun and physics: a magic square?

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Abstract

The Heun equation is a natural generalization of the hypergeometrical differential equation. Algebraic approaches to Schrödinger and/or Sturm–Liouville eigenvalue problems such as supersymmetric quantum mechanics can also be applied to the Heun equation thereby extending the set of solvable potentials considerably. The question for the underlying symmetries is addressed.

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1. Introduction

Supersymmetric quantum mechanics (SSQM) [1] exploits the properties of the time honoured factorization method [2] or, equivalently, the 'théorème curieux' by Darboux now known as Darboux transformation [3]. The factorization method has a long history in the realm of differential equations [4]. The method of Darboux transformations matured from its discovery to a versatile tool in the theory of solitons [5]. When applied to Sturm–Liouville eigenvalue problems, both procedures have been known for a long time to offer a quick test for the solvability of a given problem via the factorization condition and to find eigenvalues and eigenfunctions; they can be used to generate new (isospectral) problems from problems with known solutions, are a convenient frame for the calculation of matrix elements, allow perturbation theory [6] and can be extended to continuous spectra [7, 8]. The question if there is a deeper symmetry behind these algebraic structures (already posed by Infeld! [2]) seemed to be answered by the advent of SSQM, but the shape invariance condition raises some questions in this respect.

The so-called shape invariance condition of SSQM is considered to be a decisive test for the solvability of a given problem. Since this condition is completely equivalent to the factorization condition [9], a recent study of the symmetry underlying the shape invariance condition [10] sheds some light on the intrinsic symmetry properties of the factorization method, Darboux transformations and SSQM also. The ladder operators of the factorization method, in turn, are—at least in the case of the nonrelativistic and the relativistic hydrogen atom and the harmonic oscillator—contained in the abstract shift operators of the corresponding higher symmetry groups as sketched below for the hydrogen problem. The relations between these symmetry properties are not yet obvious [11].

Most applications of the algebraic strategies mentioned above deal in fact with the hypergeometric differential equation or confluent forms thereof. This differential equation is a Fuchsian differential equation with three regular singularities. The next member in this class is called the Heun equation, has four regular singularities and its confluent forms comprise those of the hypergeometric differential equation. Analytical solutions to the Heun equation are usually constructed on a problem by problem basis; an adaptation of SSQM techniques or its relatives to this case would find a wide range of applications as demonstrated here for some examples.

The material is organized as follows: the familiar Coulomb problem is used in the next section to demonstrate the interplay between the inherent symmetries involved as mentioned above. This is followed by a short summary of the Heun equation and its properties [12]. The physical systems then surveyed are easily tied to confluent forms of the Heun equation. The next section contains some arguments on how photonics—an area which is supposed to bloom in the current century of the photon—would benefit from a transfer of SSQM techniques. The paper is concluded by a short summary.

2. Susy, Heun and the Coulomb family

The connection between (i) Bertrand‘s theorem in classical mechanics, (ii) factorizability of the radial Schrödinger equation and (iii) the higher symmetry of the hydrogen problem/the harmonic oscillator has often been discussed in the literature. The possibility of wormholes in Bertrand‘s theorem [13] like motion in non-Euclidean spaces is still subject to discussions [11, 14]. A quantum mechanical equivalent to Bertrand‘s theorem like identifying exactly solvable central potentials with infinitely many eigenstates seems difficult to formulate since the dressing chains of the Schrödinger equation allow us to construct infinite sets of supersymmetric shape-invariant central potentials [15, 16].

The symmetry group of the nonrelativistic Coulomb problem, here defined in simplified notation as

$$H\psi = \left(\frac{1}{2}\mathbf{p}^2 - \frac{1}{r}\right)\psi = E\psi \tag{1}$$

is determined by the angular momentum $\mathbf{L}$ and the renormalized Pauli–Runge–Lenz vector $(E \neq 0)$

$$\mathbf{B} := (\operatorname{sign}(H) \times 2H)^{-1/2}\left(\frac{1}{2}(\mathbf{L}\times\mathbf{p} - \mathbf{p}\times\mathbf{L}) + \frac{\mathbf{r}}{r}\right) \tag{2}$$

where H has been defined in equation (1) and $\operatorname{sign}(H) = \mp 1$ for bound/scattering states. These operators generate the symmetry groups $SO(4)/SO(3,1)$ for the bound/scattering states. Labelling, for instance, the eigenkets of the bound states as $|nlm\rangle$, the action of the shift operators of $SO(4)$ can be used to construct shift operators for the quantum numbers (l, m) in terms of $\mathbf{L}$ and $\mathbf{B}$ reading

$$\mathbf{S}^- = \mathbf{B}(O+1) - \tfrac{1}{2}[L^2, \mathbf{B}] \qquad \mathbf{S}^+ = \mathbf{B}O + \tfrac{1}{2}[L^2, \mathbf{B}] \tag{3}$$

with

$$O = \left(L^2 + \tfrac{1}{4}\right)^{1/2} - \tfrac{1}{2}. \tag{4}$$

The superscripts in equation (3) symbolize the action of these operators on the quantum number l. The components S_3^-, S_3^+ act on the eigenkets $|nlm\rangle$ via

$$S_3^-|nlm\rangle \sim |nl-1m\rangle \qquad S_3^+|nlm\rangle \sim |nl+1m\rangle. \tag{5}$$

In spherical coordinates (r, θ, φ), the operators S_3^-, S_3^+ are seen to contain the ladder operators $A_l^{\mp}$ for the quantum number l in the factorization treatment of the hydrogen problem. The algebraic derivation of the S-matrix can proceed along the same lines either by complete analogy or by extending the factorization method from unit shifts in the quantum numbers to arbitrary shifts, i.e. to continuous spectra. The Dirac–Coulomb problem has a relativistic analogue of the Pauli–Runge–Lenz vector (2) again allowing for an algebraic derivation of eigenvalues, eigenfunctions and S-matrix and containing the corresponding ladder operators of the factorization method [7].

The isotropic harmonic oscillator with the symmetry group $SU(3)$ can be treated along the same lines tying again the generators of the symmetry group to the ladder operators of the factorization scheme [17]. No higher symmetry groups are known for all other potentials treated so far by the factorization method.

Factorizable problems can be modified by a generalization of the ladder operators

$$A_l^{\mp} = \pm\frac{\mathrm{d}}{\mathrm{d}r} + k(r, m) \tag{6}$$

where the $k(r, m)$ are determined by plugging a simple solution (ϕ_0, λ_1) of the Schrödinger equation into the factorization condition [2]

$$k^2(r, m+1) + k'(r, m+1) + k^2(r, m) - k'(r, m) = L(m) - L(m+1). \tag{7}$$

Allowing now for a solution of (7) in the form

$$\phi_c = \frac{1}{\phi_0}\left(1 + c\int_0^r \mathrm{d}r'\phi_0(\lambda_1, r')\right) \tag{8}$$

where c is an adjustable constant, the generation of a plethora of 'deformations' of the original solvable potential with a fine-tuning of the corresponding eigenvalues is possible. A change in the range of integration from 0 to ∞ also works here. All the resulting Schrödinger equations are confluent forms of Heun's equation.

The two Coulomb-centre problem provides a nice self-adjoint example of the confluent Heun equation with an interesting symmetry structure allowing separation of coordinates in prolate spheroidal coordinates. The Coulomb spheroidal functions involved can be shown to provide in the case of bound states non-canonical degenerate representations of $O(2, 2) \times O(2, 2)$, while the scattering states copy the $SO(3, 1)$ symmetry addressed above to $SO(3, 1) \times SO(3, 1)$ [12].

3. The Heun equation

Any homogeneous linear second-order differential equation with four regular singularities, i.e. any Fuchsian second-order equation, can be transformed into the canonical form of Heun's equation reading [12]

$$y''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)y'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y(z) = 0 \tag{9}$$

where y and z are complex variables, $\alpha, \beta, \gamma, \delta, \epsilon, q, a$ are arbitrary complex parameters, $a \neq 0, 1$ and the relation

$$\gamma + \delta + \epsilon = \alpha + \beta + 1 \tag{10}$$

holds. This equation has four regular singularities at $z = 0, 1, a, \infty$.

The differential equation (9) has several classes of solutions:

(i) Local solutions between singularities to be constructed by power series sometimes allowing an analytical continuation outside the original domain;
(ii) If one can find local solutions being analytic in some domain including two singularities, these solutions are coined Heun functions constructed via an eigenvalue problem for the (accessory) parameter q;
(iii) In the case where three local solutions around three different singularities coincide and the parameters satisfy certain conditions, the solutions are called Heun polynomials, have a finite closed form and can be found via a two-parameter eigenvalue problem;
(iv) Path multiplicative solutions, which, when continued analytically around a closed path surrounding two singularities, return to the starting point multiplied by a constant.

The Heun equation is a generalization of the Gauss hypergeometric differential equation with three singularities (at $z = 0, 1, \infty$). There are several ways to achieve the degeneration of the former to the latter via appropriate choices of parameters in equation (9) like [12] $a := 1, q := \alpha\beta$ or setting $a = q = 0$ or defining $\epsilon = 0, q = a\alpha\beta$. The relation between both differential equations is often used to construct new solutions to Heun's equation via solutions of the hypergeometric differential equation [18].

The close connection between the Heun equation and differential equations of physical interest accessible via SSQM is nicely demonstrated via the Lamé equation. This equation can be written in algebraic form as (cf equation (9))

$$y''(z) + \frac{1}{2}\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-a}\right) y'(z) + \frac{ah - \nu(\nu+1)z}{4z(z-1)(z-a)} y(z) = 0. \qquad (11)$$

This at a first glance though not a very appealing equation takes a familiar form via the transformations

$$a = \frac{1}{k^2} \qquad z = sn^2(x,k) y(z) = w(x) \qquad (12)$$

where $sn(x,k)$ is a Jacobi elliptic function. The transformed equation

$$w''(x) + (h - \nu(\nu+1)k^2 sn^2(x,k))w(x) = 0 \qquad (13)$$

is a form of the Lamé equation accessible via SSQM and its relatives [19].

The Heun operator $H(y(z))$ defined in equation (9) can actually be factorized in the form [20]

$$H = \left(L_1(z)\frac{\mathrm{d}}{\mathrm{d}z} + M_1(z)\right)\left(L_2(z)\frac{\mathrm{d}}{\mathrm{d}z} + M_2(z)\right) \qquad (14)$$

with polynomials $L_i(z), M_i(z)$. New Heun equations to be obtained by applying this factorization can alternatively be derived by F-homotopic transformations defined below [20].

Special choices for the parameters and coalescence of singularities in equation (9) leads to the four significant confluent cases of Heun's equation reading in the so-called normal form

$$y''(z) + I(z)y(z) = 0 \qquad (15)$$

with different invariants $I(z)$ reading:

(i) Heun's general equation:

$$I(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-a} + \frac{D}{z^2} + \frac{E}{(z-1)^2} + \frac{F}{(z-a)^2} \qquad (16)$$

(ii) Confluent Heun equation:

$$I(z) = A + \frac{B}{z} + \frac{C}{z-1} + \frac{D}{z^2} + \frac{E}{(z-1)^2} \tag{17}$$

(iii) Biconfluent Heun equation:

$$I(z) = Az^2 + Bz + C + \frac{D}{z} + \frac{E}{z^2} \tag{18}$$

(iv) Double confluent Heun equation:

$$I(z) = A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \frac{E}{z^4} \tag{19}$$

(v) Triconfluent Heun equation:

$$I(z) = Az^4 + Bz^3 + Cz^2 + Dz + E. \tag{20}$$

In each case, the parameters are not at all independent but satisfy certain conditions such as $A + B + C = 0$ in equation (16).

In the context of the hypergeometric differential equation it is well known that point canonical transformations can be used to generate new isospectral potentials or to group the solvable cases into a few classes with certain prototypes [21, 22]. In the case of the Heun equation, two types of transformations have to be distinguished:

(i) There are 24 fractional linear (Möbius) transformations of the independent variable mapping three of the four singularities $0, 1, a, \infty$ into $0, 1, \infty$, while the fourth singularity is mapped into a different point;

(ii) there are two different types of transformations of the dependent variable: F-homotopic transformations have the form

$$y(z) = z^{\rho}(z-1)^{\sigma}(z-a)^{\tau}w(z) \tag{21}$$

and map the equation for $y(z)$ into an equation for $w(z)$ upon a careful choice of ρ, σ, τ in terms of the original parameters [12]; the related s-homotopic transformations differ by an exponential term in the RHS of equation (21).

Factorization methods and/or transformations of this type can be used to generate new solvable potentials as demonstrated now considering some examples.

4. Examples

If the invariants $I(z)$ of Heun's equation defined in equations (15)–(20) have the form

$$I(z) = k^2 - V(z) \tag{22}$$

point canonical transformations (F/s -homotopic transformations) can be used to generate new families of solvable potentials. This has been done systematically in [23] obtaining several new classes of solvable potentials for the Schrödinger equation solvable in terms of Heun functions. (Many more examples, focusing on generalizations of the classical equations of mathematical physics, can be found in [12] and the references cited therein.) Familiar potentials like the hydrogen-molecule ion, the Coulomb potential, the harmonic oscillator and the Stark effect for hydrogen as well as for the harmonic oscillator have been recovered in [23] thereby embedding these systems with its well-known symmetries in the context of Heun's equation. Since point canonical transformations can be brought in the frame of SSQM [21], the results of [23] are also tractable via SSQM thereby extending the application of SSQM to Heun's equation and its confluent forms.

Some potentials contained in equations (15)–(20) have already been applied. Potentials of type

$$V(r) = \sum_{i=1}^{4} V_i r^{-\mathrm{i}} \tag{23}$$

for instance, defining a double confluent Heun equation, found applications in high energy magnetic interactions [24], solid-state physics describing diffusion and drift of kinks [25], phase transitions in certain systems [26] or in the periodical system such as Demkov–Ostrovsky potential [27].

Linear combinations of the various terms of the biconfluent/double confluent normal forms of Heun's equation can be used to generate potentials with bound states at threshold ($E = 0$) exhibiting interesting properties in classical and in quantum mechanics [28]. Denoting the background potential with V_0, the Schrödinger equation leads for $E = 0$ to

$$V_0(r) = \frac{\psi''}{\psi}. \tag{24}$$

The ansatz (real μ)

$$\psi(r) = \frac{\sin r^{\mu}}{r^{(\mu-1)/2}} \tag{25}$$

leads to the potentials

$$V(r) = \frac{\mu^2 - 1}{4r^2} - \mu^2 r^{2\mu-2}. \tag{26}$$

These potentials have the eigenvalue $E = 0$ with normalizable eigenfunctions for $\mu > 2$.

Plugging these potentials into the classical equation of motion leads to an interesting result [29]: a point particle moving in this force field escapes in a finite time to ∞, is reflected, returns to a minimal $r = r_{\mathrm{min}}$ and bounces back from there to ∞. The physical meaning of a finite travel time for an infinite distance remains to be explored.

A potential of type

$$V(r) = \frac{6r(r^3 - 2c)}{(r^3 + c)^2} \tag{27}$$

first derived in [30] via the Gel'fand–Levitan formalism, is a potential with a zero energy bound state causing no scattering at all: the Jost-matrix $F(k)$ and the phase shift $\delta(k)$ satisfy

$$F(k) = 1 \qquad \delta(k) = 0. \tag{28}$$

Another class of transparent potentials is provided by the so-called von Neumann–Wigner potentials with bound states embedded in the continuum (beic) [29]. These long-ranged oscillating potentials characterized by the asymptotics

$$\lim_{r\to\infty} V(r) = \frac{b \sin cr}{r} + O\left(\frac{1}{r^2}\right) \tag{29}$$

have an embedded eigenvalue at $k^2 = c^2/4$, iff the resonance condition

$$\frac{|b|}{|2c|} > \frac{1}{2} \tag{30}$$

holds. If a potential with this asymptotics has an embedded eigenvalue, scattering in this potential is trivial with $F(k) = 1, \delta(k) = 0$ for $|b/c| = 2n, n = 1, 2, \ldots$ [31]. Potentials with zero energy bound states such as (27) leading to trivial scattering follow from von Neumann–Wigner potentials in the limit $k^2 -> 0$ [28, 30, 31].

The transparent von Neumann–Wigner and zero-energy potentials must be distinguished from the classical reflectionless potentials with a nontrivial transmission coefficient which constitute the soliton solutions of the KdV equation [5].

Experimental verifications of embedded eigenvalues, mimicking the long-ranged asymptotics of von Neumann–Wigner potentials by finite superlattices with a defect have been reported in [32] and in the interactions between neighbouring adatoms on metal surfaces [33]. The long-ranged oscillating asymptotics are also realized in 1D Friedel oscillations caused by an impurity in a wire where the electrons experience a potential of type [33]

$$\lim_{x\to\infty} V_{\mathrm{eff}} = \frac{a\cos 2k_F x}{x} + V_{\mathrm{short}}(x) \tag{31}$$

where k_F stands for the Fermi energy.

Embedded eigenvalues are very fragile objects: the slightest perturbation like shifting the starting function $\psi(r) = 1/\kappa\,\sin(\kappa r)$ used in the Darboux formalism to generate a beic-bearing potential to $\tilde{\psi}(r) = 1/\kappa\,\sin(\kappa r - \alpha)$ suffices to transform the bound state into a Breit–Wigner shaped resonance with nontrivial scattering [31, 34]. Similar instabilities can be found in the case of embedded eigenvalues of the Helmholtz equation [35]. Screened beic-bearing potentials such as $V(r) \sim \exp(ar)\cos(br)/r$ or the truncation of a beic-bearing potential, by comparison, lead to different resonance shapes and cross sections [36].

The notion of transparent potentials used here differs from the transparency of Newton–Sabatier potentials: these long-ranged oscillating potentials decay like $r^{-3/2}$, do not have embedded eigenvalues and are only transparent in the Born approximation [37].

In the theory of nonlinear evolution equations, the 1D Schrödinger equation appears as one equation of the Lax pair of the Korteweg–de Vries (KdV) equation, i.e. as a Sturm–Liouville eigenvalue problem containing the solution of the KdV equation as a potential. This provides an application of Heun's equation in the theory of solitons. Many of the solvable potentials discussed so far in the context of quantum mechanics actually define special solutions of the KdV equation. The rational solutions of the KdV equation [5, 38], for instance, realize confluent forms of Heun's equation and the corresponding one-dimensional Schrödinger equations can be treated via SSQM. The singularities arising in course of calculation lead to point singularities of type g/x^2, which are apparently only penetrable for $g \in [-0.25, 0.75]$ and splitting the axis in all other cases [39]. This complication does not occur in the case of the one-dimensional Helmholtz equation, where the reflectionless potentials contain singularities of type $1/(x-x_i)^2$ [40]. The one-dimensional analogue of von Neumann–Wigner potentials are long-ranged oscillating solutions of the KdV equation coined positon solutions. Since positons correspond formally to degenerate eigenvalues in one dimension, the resulting solutions are necessarily singular. The concept of positons (and degenerate soliton solutions coined negatons) can be extended to any solvable nonlinear evolution equation.

A last example to be mentioned here are effective mass problems, i.e. Schrödinger equations with a position-dependent mass $m(x)$. For a smooth potential and mass step reading

$$V(x) = \frac{V_0}{2}\left(1 + \tanh\frac{x}{2x_0}\right) \qquad m(x) = \frac{m_1+m_2}{2} + \frac{m_1-m_2}{2}\tanh\frac{x}{2x_0} \tag{32}$$

one obtains already upon a simple transformation of the independent variable the general Heun equation (9) [41]. Many other effective mass problems are known by now [42]. The SSQM techniques used to solve the corresponding Schrödinger equations are in fact solutions of various confluent forms of Heun's equation.

5. Susy and Heun in Photonics

The Y2K marked the start of the century of the photon with high claims about the goals to be achieved in the near future as an inspection of the WWW shows. In this context, general Helmholtz equations with spatially dependent dielectricity/permeability functions will be encountered very often. The effective mass problems of quantum mechanics mentioned above provide a nice way to exploit techniques of SSQM in photonics despite the different dispersion relations. The Schrödinger equation

$$-\frac{\hbar}{2}\nabla\cdot\left(\frac{1}{m(\vec{r})}\nabla\psi(\vec{r})\right)+V(\vec{r})\psi(\vec{r})=E\psi(\vec{r}) \tag{33}$$

with the position-dependent mass $m(\vec{r})$ obviously has a structure comparable to the Helmholtz equation

$$-\nabla\cdot\left(\frac{1}{\mu_r(\vec{r})}\nabla\vec{E}\right)+k_0^2\epsilon_r(\vec{r})\vec{E}=0. \tag{34}$$

When assuming for definiteness TE-polarization for the incoming beam (TM analogous), the quantities in the Helmholtz equation take the form

$$\mu_r(\vec{r})=\frac{\mu(\vec{r})}{\mu_0}\qquad\epsilon_r(\vec{r})=\frac{\epsilon(\vec{r})}{\epsilon_0}\qquad k_0^2=\frac{\omega^2}{c^2}\mu_0\epsilon_0. \tag{35}$$

The close analogies between optics and quantum mechanics for one-particle systems [43] allow us to translate solutions of the Schrödinger equation into the corresponding solutions of the Helmholtz equation. (Energy entangled particles, in comparison, behave differently for matter waves and for light waves [44]). The quantum mechanical effective mass problems mentioned above [41, 42] can be solved by means of SSQM. A translation of these techniques and solutions into the realm of photonics governed by the corresponding Helmholtz equation opens a new market for the benefit of both sides, quantum mechanics and optics. First attempts in this direction concern supersymmetric methods for the index of refraction in some special cases or light transmission through inhomogeneous optical waveguides having shape-invariant index of refraction profiles [45].

Moreover, an application of algebraic methods sometimes simplifies the calculation considerably as two examples show: (i) the Helmholtz equation for the propagation of the TM mode in an inhomogeneous slab guide can be transformed into a harmonic oscillator equation and subsequently solved by factorization [46]; (ii) the Helmholtz equation for a refractive index profile in the form of the mass step defined above leads like in the quantum mechanical case to a Heun equation of type (9) and can thus be solved in a straightforward way [47].

6. Summary and outlook

Heun's equation is a natural generalization of the Gauss hypergeometric differential equation. It has been demonstrated that Heun's equation and/or its confluent forms arise in many physical systems. In the case where the differential equation describing the system at hand is identified as being of Heun type, the corresponding mathematical formalism can be used to find its solutions. Moreover, in the theory of Heun's equation, many solvable potentials have been derived in the past still awaiting physical systems.

The examples of zero-energy potentials and von Neumann–Wigner type potentials have been used to show that Heun's equation and its confluent forms can be treated by algebraic methods such as SSQM. A generalization of the shape invariance/factorization condition from the hypergeometric differential equation to Heun's equation has not been achieved so far.

The analogies between optics and quantum mechanics allow us to export the concepts of SSQM to photonics. Several systems (including Heun's equation) have already been identified where quantum mechanical methods can be used successfully to solve optical problems described by the Helmholtz equation. A physical interpretation of the structure of SSQM in optics remains to be developed.

Other areas of applications awaiting exploration by SSQM techniques not included here due to the lack of space should at least be mentioned: (i) the Rayleigh equation, used to study the Rayleigh–Taylor instability of fluids in gravitational fields, can be mapped on the Schrödinger equation [48]; (ii) generalizations of the Fokker–Planck equation have been discussed recently leading directly to the general Heun equation (9) [49]. Factorizations thereof such as those discussed in [20] might facilitate the search for solutions of the generalized Fokker–Planck equation considerably.

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References

[1] Witten E 1981 *Nucl. Phys.* B **188** 113
[2] Infeld H and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[3] Darboux G 1882 *C. R. Acad. Sci. Paris* **94** 1456
[4] Schlesinger L 1895 *Handbuch der Theorie der Linearen Differentialgleichungen* (Leipzig: Teubner)
Ince E L 1927 *Ordinary Differential Equations* (London: Longmans, Green and Co.)
[5] Matveev V B and Sall' M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[6] Bessis N and Bessis G 1987 *J. Phys. A: Math. Gen.* **20** 5745
[7] Stahlhofen A A 1997 *Helv. Phys. Acta* **70** 372
[8] Eastham M S P and Kalf H 1982 *Schrödinger-Type Operators with Continous Spectra* (*Pitman Research Notes in Mathematics* vol 65) (London: Pitman)
[9] Stahlhofen A A 1989 *J. Phys. A: Math. Gen.* **22** 1053
[10] Faux M and Spector D 2004 A BPS interpretation of shape invariance *Preprint* quant-ph/0401163v1
[11] McIntosh H V 2004 *Symmetry and Degeneracy* www.delta.cs.cinvestav.mx/mcintosh/comun/symm/symm.pdf
[12] Ronveaux A (ed) 1995 *Heun's Differential Equations* (Oxford: Oxford Science Publications)
[13] Bertrand J 1873 *C. R. Acad. Sci., Paris* **77** 849
[14] Martinez-y-Romero R P, Nunez-Yepez H N and Salas-Brito A L 2002 *Am. J. Phys.* **70** 1059
[15] Novokshenko V Yu 1995 *Physica* D **87** 109
[16] Lipan O and Rasinariu C 2002 *J. Math. Phys.* **43** 847
[17] DeLange O L and Raab R E 1991 *Operator Methods in Quantum Mechanics* (Oxford: Oxford University Press)
[18] Ishkhanyan A and Suominen K-A 2003 *J. Phys. A: Math. Gen.* **36** L81
Gaspar Y 2003 New transformation law for Heun and hypergeometric equations *Preprint* math-ph/0305050v1
Maier R S 2002 Transforming the Heun equation to the hypergeometric equation: I. Polynomial transformations math.CA/0203264v1
[19] Sukhatme U and Khare A 2002 *Phys. At. Nucl.* **65** 1122
[20] Ronveaux A 2003 *Appl. Math. Comput.* **14** 177
[21] Lévai G 1993 Solvable potentials derived from supersymmetric quantum mechanics *Quantum Inversion Theory and Applications Springer Lecture Notes in Physics* vol 427 ed H V Geramb (Berlin: Springer) p 107
[22] Gangopadhyaya A *et al* 1994 *Helv. Phys. Acta* **67** 363
[23] Lemieux A and Bose A K 1969 *Ann. Inst. Henri Poincare* **10** 259
[24] Aly H H and Barut A O 1990 *Phys. Lett.* A **145** 299
[25] Seeger A 1990 Diffusion with drift and relationships to confluent Heun equations *Centennial Workshop on Heun's Equation—Theory and Applications* ed A Seeger and W Lay (Stuttgart: Max-Planck-Institut für Metallforschung)
[26] Zia R K P *et al* 1988 *Am. J. Phys.* **56** 160

[27] Rosu H C *et al* 1995 *Phys. Lett.* A **208** 33
[28] Alhaidari A D 2002 *Int. J. Mod. Phys.* A **17** 4551
[29] Von Neumann J and Wigner E 1929 *Phys. Z.* **48** 465
[30] Moses H E and Tuan S F 1959 *Nuovo Cimento.* **13** 446
[31] Stahlhofen A A 1996 *J. Phys. A: Math. Gen.* **29** L581
[32] Capasso F *et al* 1992 *Nature* **358** 565
Zahler M *et al* 1992 *Appl. Phys. Lett.* **61** 949
[33] Stahlhofen A A and Druxes H 1999 *Phys. Lett.* A **253** 214
[34] Moses H E 1971 *Stud. Appl. Math.* **58** 187
Pursey D L and Weber T A 1995 *Phys. Rev.* A **52** 3932
[35] Malykh M D 2002 *Moscow Univ. Phys. Bull.* **57** 17
[36] Weber T A and Pursey D L 1998 *Phys. Rev.* A **57** 3534
[37] Chadan K and Sabatier P C 1989 *Inverse Problems in Quantum Scattering Theory* 2nd edn (Berlin: Springer)
[38] Ohmiya M 1988 *Osaka J. Math.* **25** 607
[39] Bulla W and Gesztesy F 1988 *J. Math. Phys.* **26** 2520
[40] Novikov V S 2000 *JETP Lett.* **3** 153
[41] Dekar Liès *et al* 1999 *Phys. Rev.* A **59** 107
[42] De Souza Dutra A and Almeida C A S 2003 Exact solvability of potentials with spatially dependent effective mass *Preprint* quant-ph/0306065v1
[43] Hupert J J 1975 *Appl. Phys.* **6** 131
[44] Brukner C and Zeilinger A 1997 *Phys. Rev. Lett.* **79** 2599
[45] Chumakov S M and Wolf K B 1994 *Phys. Lett.* A **193** 51
Rostami A and Moyaedi S K 2001 *Ind. J. Phys.* B **75** 357
[46] Rostami A and Moyaedi S K 2003 *J. Opt. A: Pure Appl. Opt.* **5** 380
[47] Van Duis C A and Slujiter F W 1980 *Radio Sci.* **15** 95
[48] Mikaelian K O 1996 *Phys. Rev.* E **53** 3551
[49] Debosscher A 1998 *Phys. Rev.* E **57** 252